Difference and Integral Calculus on Weighted Networks

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The discrete vector calculus theory is a very fruitful area of work in many mathematical branches not only for its intrinsic interest but also for its applications, [1, 2, 3, 4]. One can construct a discrete vector calculus by considering simplicial complexes that approximates locally a smooth manifold and then use the Whitney application to define inner products on the cochain spaces, which gives rise to a combinatorial Hodge theory. Alternatively, one can approximate a smooth manifold by means of non-simplicial meshes and then define discrete operators either by truncating the smooth ones or interpolating on the mesh elements. This approach is considered in the aim of *mimetic methods* which are used in the context of difference schemes to solve numerically boundary values problems. Finally, another approach is to deal with the mesh as the unique existent space and then the discrete vector calculus is described throughout tools from the Algebraic Topology since the geometric realization of the mesh is a unidimensional CW-complex.

Our work falls within the last ambit but, instead of importing the tools from Algebraic Topology, we construct the discrete vector calculus from the graph structure itself following the guidelines of Differential Geometry. The key to develop our discrete calculus is an adequate construction of the tangent space at each vertex of the graph. The concepts of discrete vector fields and bilinear forms are a likely result of the definition of tangent space. Moreover, we obtain discrete versions of the derivative, gradient, divergence, curl and Laplace-Beltrami operators that satisfy the same properties that its continuum analogues. We also introduce the notion of order of an operator that recognizes the Laplace-Beltrami operator as a second order operator, while the rest of the above-mentioned operators are of first order. Also we construct the De Rham cohomology of a weighted networks, obtaining in particular a Hodge decomposition theorem type. On the other hand we develop the corresponding integral calculus that includes the discrete versions of the Integration by Parts technique and Green's Identities. As an application we study the variational formulation for general boundary value problems on weighted networks, obtaining in particular the discrete version of the Dirichlet Principle.

[1] E. Bendito, A. Carmona and A. M. Encinas, Difference schemes on uniform grids performed by general discrete operators, *Appl. Num. Math.*, **50** (2004), 343-370.

[2] A. Bensoussan and J.L. Menaldi, Difference equations on weighted graphs, *J. Convex Anal.*, **12** (2005), 13-44.

[3] S.Y. Chung and C.A. Berenstein, ω-harmonic functions and inverse conductivity problems on networks. *SIAM J. Appl. Math.* **65** (2005), 1200–1226.

[4] W. Schwalm, B. Moritz, M. Giota and M. Schwalm, Vector difference calculus for physical lattice models, *Phy. Rev. E* **59** (1999), 1217-1233.